The WKB approximation in the deformed space with the minimal length and minimal momentum

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Abstract

A Bohr-Sommerfeld quantization rule is generalized for the case of the deformed commutation relation leading to minimal uncertainties in both coordinate and momentum operators. The correctness of the rule is verified by comparing obtained results with exact expressions for corresponding spectra.

Keywords: deformed Heisenberg algebra, Bohr-Sommerfeld quantization rule.

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1 Introduction

Several independent lines of theoretical physics investigations (e.g. string theory and quantum gravity) suggest the existence of finite lower bound to the possible resolution of length ΔX (minimal length) [1–3]. Such an effect can be achieved by modifying the usual canonical commutation relations [4–7].

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In our previous article [8], we have developed a semiclassical approach for the deformed space with the following commutation relation between the coordinate $\hat{X}$ and the momentum $\hat{P}$ operators:

$$[\hat{X}, \hat{P}] = i\hbar f(\hat{P}), \tag{1}$$

where f is a positive function of the momentum. Such a deformation may lead to the existence of the minimal length. Using the semiclassical analysis, we have shown that in the deformed space usual Bohr-Sommerfeld quantization rule for canonically conjugated variables, x and p,

$$\oint p dx = 2\pi\hbar(n + \delta) \tag{2}$$

is valid [8]. This expression was successfully used to consider a one-dimensional Coulomb-like problem [9]. Parameter δ depends on boundary conditions: for smooth potentials and deformation function $f(P)$ such that $f(0) \neq 0$, it equals $1/2$. The rule (2) can be rewritten in terms of classical variables X and P corresponding to the initial coordinate and momentum operators:

$$-\oint \frac{X dP}{f(P)} = 2\pi\hbar(n + \delta). \tag{3}$$

2 Bohr-Sommerfeld quantization rule

The purpose of this paper is to generalize result (3) for a wider class of the deformed commutation relation

$$[\hat{X}, \hat{P}] = i\hbar f(\hat{X}, \hat{P}), \tag{4}$$

where f is a positive function of the position and momentum. Such a modification of the canonical commutation relation can lead to the existence of nonzero minimal uncertainties of the position and the momentum operators. Function,

$$f(\hat{X}, \hat{P}) = 1 + \alpha\hat{X}^2 + \beta\hat{P}^2, \tag{5}$$

is an example of the deformation leading to the minimal length and minimal momentum. In this case $\Delta X = \hbar\sqrt{\beta/(1 - \hbar^2\alpha\beta)}$, $\Delta P = \hbar\sqrt{\alpha/(1 - \hbar^2\alpha\beta)}$. We use only this deformation function in this paper, considering examples in the third section.

Let us know the representation of operators $\hat{X}$ and $\hat{P}$ satisfying (4)

$$\hat{X} = X(\hat{x}, \hat{p}), \qquad \hat{P} = P(\hat{x}, \hat{p}), \tag{6}$$

where small operators $\hat{x}$, $\hat{p}$ are canonically conjugated. It seems that this representation can be found formally as series over deformation parameters (for examples of such expansion see [10]). The explicit representation of $\hat{X}$ and $\hat{P}$ operators for deformation function (5) can be found in [11]. Note that when the deformation function depends only on the position, i. e. $f = f(X)$, the commutation relation describes a particle with a position-dependent mass $M(X) = 1/f^2(X)$ [12].

In the classical limit $\hbar \to 0$ the deformed commutation relation (4) leads to the deformed Poisson bracket [13, 14]

$$\{X, P\} = f(X, P). \tag{7}$$

It is always possible to choose such canonically conjugated variables x and p (Darboux theorem [15]) that X and P as functions of x, p satisfy

$$\{X, P\}_{x,p} = \frac{\partial X}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial x}\frac{\partial X}{\partial p} = f(X, P). \tag{8}$$

We consider that $P(x, p)$ is an odd monotonic function with respect to p and even function of x; $X(x, p)$ is an odd monotonic function with respect to x and an even function of p. This provides left-right symmetry and one-to-one correspondence between P and p, X and x, respectively. The one-dimensional Schrödinger equation in the deformed space reads

$$\frac{\hat{P}^2}{2m}\psi + U(\hat{X})\psi = E\psi. \tag{9}$$

Let us write the wave function in the following form

$$\psi(x) = \exp\left[\frac{i}{\hbar}S(x)\right]$$

then in the linear approximation over $\hbar$

$$H\left(x, -i\hbar\frac{d}{dx}\right)\psi(x) = \left[H(x, S'(x)) - \frac{i\hbar}{2}H_{pp}(x, S'(x))S''(x) + \dots\right]\psi(x), \tag{10}$$

where subscript p denotes derivative with respect to the second argument of $H(x, p)$ and the prime denotes derivative with respect to

x. Formula (10) is derived for a normally ordered Hamiltonian (the powers of operator $-i\hbar\frac{d}{dx}$ act on the wavefunction first and then are multiplied by the functions of x). Expanding $S(x)$ in power series over $\hbar$

$$S(x) = S_0(x) + \frac{\hbar}{i}S_1(x) + \dots \tag{11}$$

from (10) we obtain the following set of equations for S:

$$H\left(x, S_0'(x)\right) = E, \tag{12}$$

$$H_p\left(x, S_0'(x)\right)\frac{\hbar}{i}S_1'(x) - \frac{i\hbar}{2}H_{pp}\left(x, S_0'(x)\right)S_0''(x) = 0. \tag{13}$$

From the requirement that $P(x,p)$ is an odd monotonic function with respect to p and the Hamiltonian form $H = \frac{P^2}{2m} + U(X)$ we obtain that equation (12) has only two solutions with respect to $S_0'(x)$. They differs only by sign, $S_0'(x) = \pm p(x)$. Then equation (13) can be rewritten as follows

$$H_p(x, \pm p)S_1'(x) = \mp\frac{1}{2}p' H_{pp}(x, \pm p)$$

and $S_1(x)$ can be found from it. Note, that since $H(x,p)$ is an even function with respect to p, S_1 is also an even function of p and thus does not depend on $S_0(x)$ sign. Then the wavefunction in the linear approximation over $\hbar$ is

$$\psi(x) = \exp\left[S_1(x)\right]\left(C_1\exp\left[\frac{i}{\hbar}\int^x pdx\right] + C_2\exp\left[-\frac{i}{\hbar}\int^x pdx\right]\right). \tag{14}$$

To obtain an expression of the Bohr-Sommerfeld quantization rule, we have to analyze the behavior of wavefunction (14) at the infinities and consider matching conditions near the turning points. The obtained expression (14) is similar to the corresponding expression of non-deformed quantum mechanics [16]. Performing the analogous analysis, we obtain for bound states the same expression for the Bohr-Sommerfeld quantization condition

$$\int_{x_1}^{x_2} pdx = \pi\hbar(n+\delta), \quad n = 0, 1, 2, \dots \tag{15}$$

where x_1 and x_2 are the turning points satisfying equation $U(x) = E$, δ depends on boundary conditions and properties of $S_1(x)$. But it

is simpler to find the value of δ from the limiting procedure to the non-deformed case. Such a procedure is illustrated with the help of examples. We also would like to recall that for large quantum numbers exact δ value is not significant.

Condition (15) has the same form as in non-deformed quantum mechanics only for auxiliary canonically conjugated variables. It is possible to rewrite it in the initials variables X and P. Rewriting expression (15) in more convenient form

$$\int\limits_{H\leq E_n} dpdx = 2\pi\hbar(n+\delta) \tag{16}$$

and noting that the Jacobian of changing from x and p variables to the initial variables X and P has the form $\dfrac{\partial(x,p)}{\partial(X,P)} = \dfrac{1}{\frac{\partial(X,P)}{\partial(x,p)}} = \dfrac{1}{f(X,P)}$, we can conclude that

$$\int\limits_{H(P,X)\leq E_n} \frac{dXdP}{f(X,P)} = 2\pi\hbar(n+\delta). \tag{17}$$

3 Examples

In this section, we consider two eigenvalue problems, namely, the harmonic oscillator and the potential well. We analyze them for the following deformation of the canonical commutation relation

$$[\hat{X},\hat{P}] = i(1+\alpha\hat{X}^2+\beta\hat{P}^2). \tag{18}$$

Such a deformation was considered for the first time in [4] and it leads to the existence of nonzero minimal uncertainties of the position and the momentum operators.

Let us consider the harmonic oscillator eigenvalue problem

$$(\hat{P}^2+\hat{X}^2)\psi = E\psi. \tag{19}$$

The Bohr-Sommerfeld quantization rule (17) gives

$$E_n = \frac{(\sqrt{\alpha}+\sqrt{\beta})^2}{4\alpha\beta}e^{2(n+\delta)\sqrt{\alpha\beta}} + \frac{(\sqrt{\alpha}-\sqrt{\beta})^2}{4\alpha\beta}e^{-2(n+\delta)\sqrt{\alpha\beta}} - \frac{\alpha+\beta}{2\alpha\beta}. \tag{20}$$

Let us compare this result with exact one obtained in [11] which can be written in the following form

$$E_n = aq^n + b + cq^{-n}, \tag{21}$$

where $q=\frac{1+\sqrt{\alpha\beta}}{1-\sqrt{\alpha\beta}}\approx e^{2\sqrt{\alpha\beta}}$; a, b, c are coefficients depending on the deformation parameters α and β in a certain cumbersome way. For small α and β, these coefficients can be approximated as follows:

$$
\begin{aligned}
a &= \frac{(\sqrt{\alpha}+\sqrt{\beta})^2}{4\alpha\beta}\left(1+\sqrt{\alpha\beta}\right)+o(1),\\
b &= -\frac{\alpha+\beta}{2\alpha\beta}+o(1),\\
c &= \frac{(\sqrt{\alpha}-\sqrt{\beta})^2}{4\alpha\beta}\left(1-\sqrt{\alpha\beta}\right)+o(1).
\end{aligned}
\tag{22}
$$

The leading terms of a, b and c coincide with the corresponding coefficients in expression (20). We would like to stress that the Bohr-Sommerfeld quantization rule (17) provides the exponential dependence of the spectrum on the quantum number, n, as it should be according to the exact result (21).

In linear approximation over deformation parameters, the WKB approximation (20) for $\delta=\frac{1}{2}$ (we choose this value of δ to obtain an undeformed spectra $E_n=2n+1$ if $\alpha=\beta=0$) gives

$$
E_n = 2n+1+(\alpha+\beta)\left(n+\frac{1}{2}\right)^2. \tag{23}
$$

While the exact expression for the spectrum differs from it by $\frac{1}{4}(\alpha+\beta)+o(\alpha,\beta)$.

The second example is the potential well described by Hamiltonian

$$
\hat{H}=\hat{P}^2, \tag{24}
$$

where the coordinate X is considered to be bounded in the interval $[-a,a]$. Then Bohr-Sommerfeld quantization rule reads

$$
\int\limits_{-a}^{a}\int\limits_{-\sqrt{E_n}}^{\sqrt{E_n}}\frac{dXdP}{1+\alpha X^2+\beta P^2}=2\pi(n+\delta). \tag{25}
$$

We failed to integrate this expression exactly, thus we restrict ourselves to consider only the linear approximation over deformation parameters. For energy levels we obtain

$$
E_n=\left(\frac{\pi n}{2a}\right)^2\left[1+\frac{2}{3}\alpha a^2+\frac{2}{3}\beta\left(\frac{\pi n}{2a}\right)^2\right]+o(\alpha,\beta). \tag{26}
$$

Here we fixed the value of δ as 0 to reproduce the spectrum of the potential well in the undeformed case ($\alpha = \beta = 0$).

The Bohr-Sommerfeld quantization rule (17) predicts that the potential well (24) has only a finite amount of bound states. To show this let us consider the case of large energies $E_n \to \infty$. Since

$$\int_{-a}^{a}\int_{-\infty}^{\infty}\frac{dXdP}{1+\alpha X^2+\beta P^2} = \frac{2\pi}{\sqrt{\alpha\beta}}\text{arcsinh}\sqrt{\alpha}a$$

is finite, equation (25) has no solution for sufficiently large n. The maximal value of n for which it can be solved $n_{max} = [\text{arcsinh}\sqrt{\alpha}a/\sqrt{\alpha\beta}]$, where $[\ldots]$ denotes the integer part. In limit $\alpha \to 0$ we obtain $n_{max} = [a/\sqrt{\beta}]$. This number coincides with the amount of bound states obtained in [7] for the potential well with $\alpha = 0$. For $\alpha = 0$ the eigenvalue problem was considered in [7,8,17] and expression (26) agrees with the results of these papers.

In the case of $\beta = 0$, integral (25) can be calculated exactly and

$$E_n = \left(\frac{\pi n\sqrt{\alpha}}{2\arctan(\sqrt{\alpha}a)}\right)^2. \tag{27}$$

An eigenvalue problem corresponding to Hamiltonian (24) and the case $\beta = 0$ can be easily solved exactly and the spectrum coincides with expression (27). It is interesting to note that in the limit $a \to \infty$, we effectively obtain a free particle with a position-dependent mass and discrete spectrum for it: $E_n = \alpha n^2$.

4 Conclusions

In the paper, we consider the Bohr-Sommerfeld quantization rule for the case when the right hand of commutation relation depends both on the coordinate and the momentum operators. To validate the obtained Bohr-Sommerfeld quantization rule, we analyze two examples with its help and compare the obtained results with already known ones.

The case of exactly solvable harmonic oscillator with the deformation $f(\hat{X},\hat{P}) = 1 + \alpha\hat{X}^2 + \beta\hat{P}^2$ shows that the Bohr-Sommerfeld quantization rule is applicable for small values of the deformation parameters α, β. The second example is related to the potential well problem which has not been solved exactly in the deformed case for

arbitrary α and β. The spectrum obtained with the help of Bohr-Sommerfeld quantization rule agrees either with the already known spectrum for $\alpha = 0$ or for easily evaluated spectrum with $\beta = 0$.

This agreement is a good reason to consider that the Bohr-Sommerfeld quantization rule (17) is valid for small values of the deformation parameters and can be used as an approximate method for the spectrum estimation.

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